Tetrahedron and 3D reflection equations from quantum cluster algebras

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1. Tetrahedron and 3D reflection equations

2. New solution

3. Derivation from quantum cluster algebra

4. Tetrahedron equality as duality

5. Outlook

Reference (appeared on arXiv this morning!)

R. Inoue, AK, Y. Terashima,

Quantum cluster algebras and 3D integrability: Tetrahedron and 3D reflection equations.
math.QA 2310.14493 Fock-Goncharov quiver (Today's talk)

Tetrahedron equation and quantum cluster algebras
math.QA 2310.14529 Square quiver (Talk at BIMSA conference RTISART-23 in July.)

1. Tetrahedron and 3D reflection equations

(3D analogue of the Yang-Baxter and reflection eqs.)

Tetrahedron eq. [A.B. Zamolodchikov 80]

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124} \quad \text{on } V^{\otimes 6}$$

$$R_{ijk} \in \mathrm{End}(\overset{i}{V} \otimes \overset{j}{V} \otimes \overset{k}{V})$$

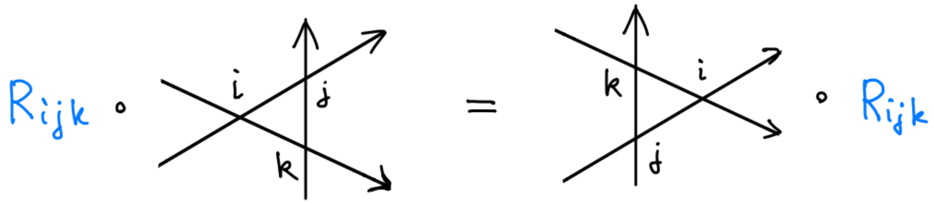
3D reflection eq. [Isaev-Kulish 97]

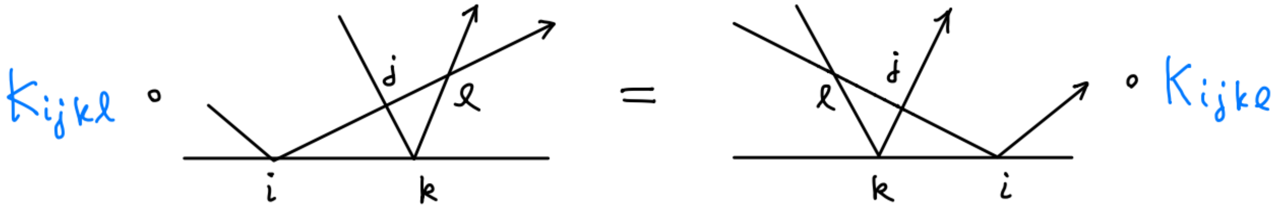
$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}$$

$$\text{on } W \otimes V \otimes W \otimes V \otimes V \otimes V \otimes W \otimes V \otimes V$$

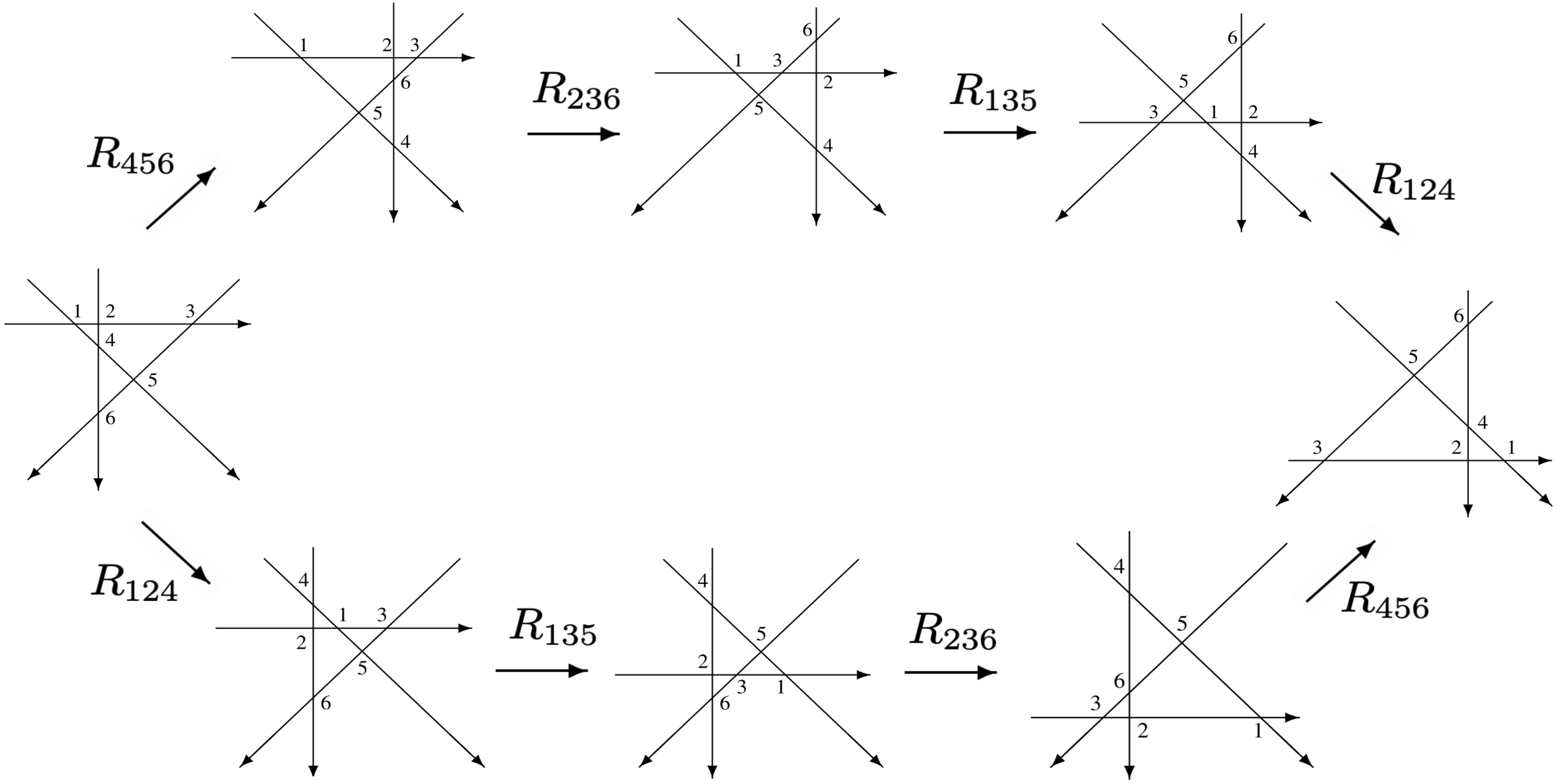
$$K_{ijkl} \in \mathrm{End}(\overset{i}{W} \otimes \overset{j}{V} \otimes \overset{k}{W} \otimes \overset{j}{V})$$

They are compatibility conditions of the quantized Yang-Baxter eq. and quantized reflection eq., which are the *usual* Yang-Baxter and reflection equations up to conjugation.



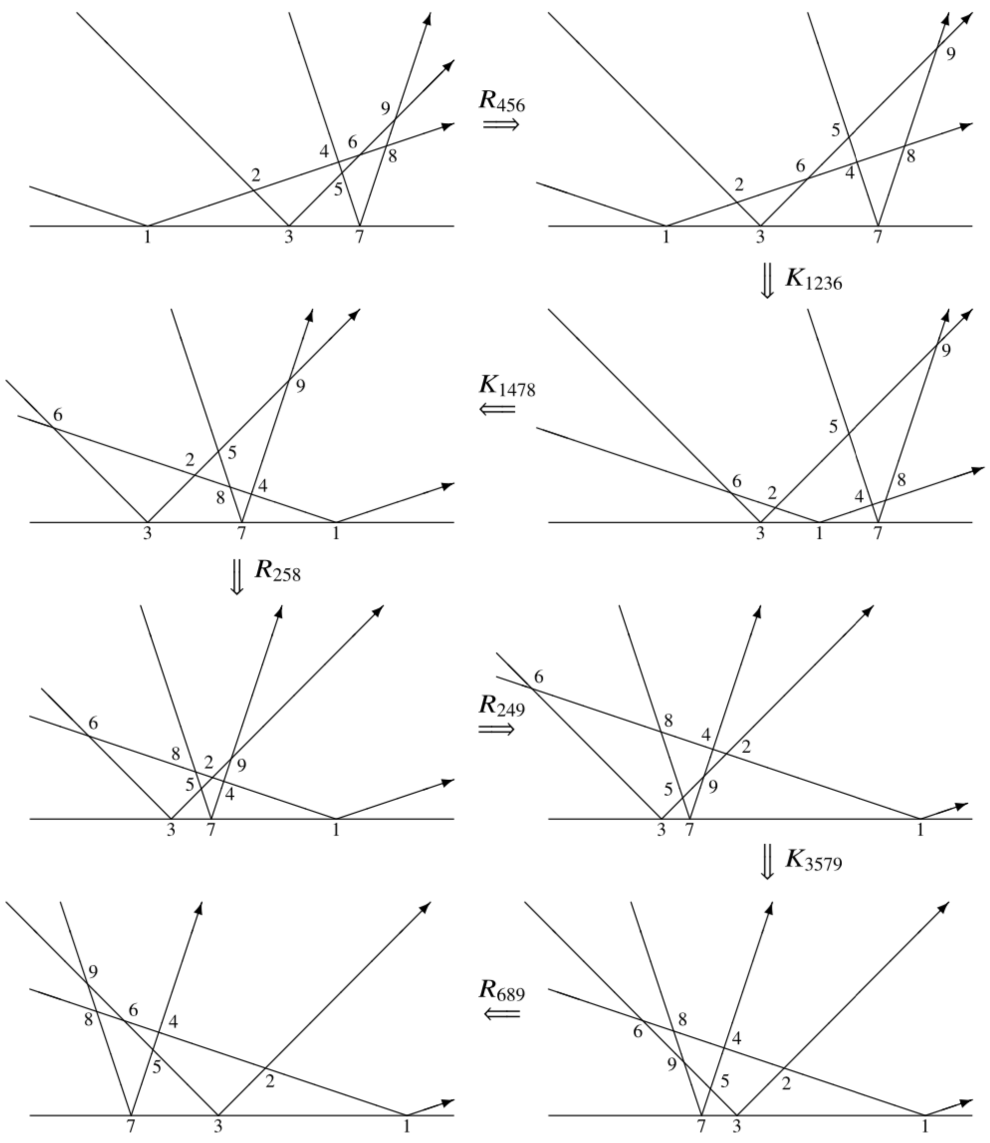


Now that R and K play the role of *structure constants*, they have to satisfy the compatibility condition under introducing one more arrow:

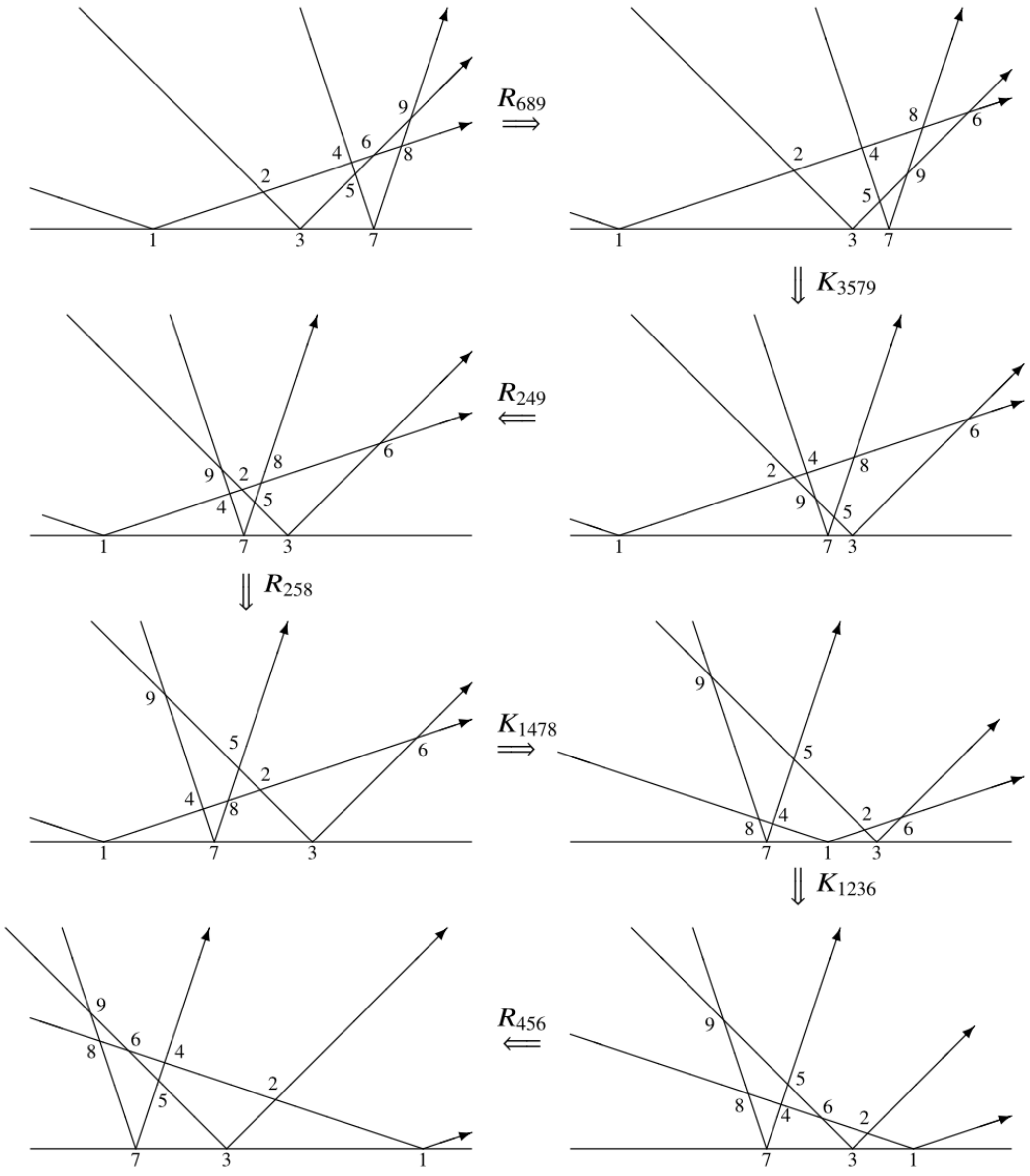


$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}$$

LHS



RHS



Several interesting solutions are known for the tetrahedron equation by Zamolodchikov, Baxter, Kapranov-Voevodsky, Bazhanov, Kashaev, Korepanov, Maillet, Mangazeev, Sergeev, Stroganov, Bytsko-Volkov, K-Matsuike-Yoneyama, etc.

Only a few solutions are known for the 3D reflection equation by K-Okado, Yoneyama.

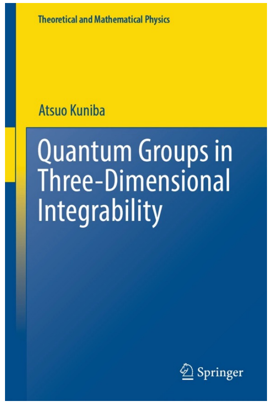
There are quantum group theoretical approaches based on quantized coordinate rings by [Kapranov-Voevodsky 94] and PBW basis of U_q^+ by [Sergeev 08].

They are equivalent beyond type A [K-Okado-Yamada 13] and have been developed extensively with many applications.

In the approach, the diagrams in the previous pages emerge as wiring diagrams for the reduced expressions of the longest element of the Weyl groups A_3 and C_3.

The aim of this talk is to develop another approach [Sun-Yagi 22], where these diagrams are complemented by quivers that facilitate the efficient operation of quantum cluster algebras.

We focus on the Fock-Goncharov quivers, devise a new realization of quantum Y-variables using q-Weyl algebras, and obtain a new solution.



2. New solution

$$\mathcal{R}_{ijk} = \Psi_q\left(e^{p_i+u_i+p_k-u_k-p_j+\lambda_{ik}}\right)\rho_{jk}\, e^{\frac{1}{\hbar}p_i(u_k-u_j)} e^{\frac{\lambda_{jk}}{\hbar}(u_k-u_i)},$$

$$\mathcal{K}_{ijkl} = \Psi_{q^2}\left(e^{p_j+u_j+p_l-u_l-2p_k+\lambda_{jl}}\right)\Psi_q\left(e^{p_i+u_i+p_k-u_k-p_j+\lambda_{ik}}\right)\Psi_{q^2}\left(e^{p_j+u_j+p_l-u_l-2p_k+\lambda_{jl}}\right)^{-1}$$
$$\times\, \rho_{jl}\, e^{\frac{1}{\hbar}p_i(u_l-u_j)} e^{\frac{\lambda_{jl}}{2\hbar}(2u_k-2u_i+u_l-u_j)}.$$

$$\Psi_q(X) = \frac{1}{(-qX;q^2)_\infty} : \text{ quantum dilogarithm}$$

Key properties

$$\Psi_q(q^2U)\Psi_q(U)^{-1} = 1 + qU,$$
$$\Psi_q(U)\Psi_q(W) = \Psi_q(W)\Psi_q(q^{-1}UW)\Psi_q(U) \text{ if } UW = q^2WU \quad \text{(pentagon identity)}$$

$$[p_i,u_j] = \begin{cases} 2\delta_{ij}\hbar & i,j\in\{3,6,9\} \\ \delta_{ij}\hbar & \text{otherwise} \end{cases} \left(\begin{matrix} [p_i,u_j]=\delta_{ij}\hbar \\ \text{for tetrahedron eq.} \end{matrix}\right) \quad [p_i,p_j]=[u_i,u_j]=0 : \text{ canonical variables}$$

$$\rho_{ij} = \text{transposition } p_i \leftrightarrow p_j,\ u_i \leftrightarrow u_j \qquad q = e^{\hbar},\ \lambda_{ij} = \lambda_i - \lambda_j$$

3. Derivation from quantum cluster algebra

$\mathrm{Seed} = (B, \mathbf{Y})$

$B \leftrightarrow Q$: quiver with vertices $1, \dots, n$

$B = (b_{ij})_{i,j=1}^{n}, \ \ b_{ij} = -b_{ji} \in \mathbb{Z}/2$: Exchange matrix (Type A only)

$\mathbf{Y} = (Y_1, \dots, Y_n), \quad Y_i Y_j = q^{2b_{ij}} Y_j Y_i$: Y-variables

$b_{ij} = 1$: $i \longrightarrow j$

$\mathbb{F}(\mathbf{Y}) = \mathbb{F}(B, \mathbf{Y})$: non-commutative fraction field generated by $\mathbf{Y}$

$b_{ij} = 1/2$: $i \cdots\cdots\rightarrow j$

Mutation

$$\mu_k(B, \mathbf{Y}) = (B', \mathbf{Y}') \qquad k \in \{1, \dots, n\}$$

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + [b_{ki}]_+ b_{kj} + [b_{kj}]_+ b_{ik} & \text{otherwise} \end{cases} \qquad [x]_+ = \max(x, 0)$$

$$Y'_i = \begin{cases} Y_k^{-1} & i = k \\ q^{b_{ik}[b_{ki}]_+} Y_i Y_k^{[b_{ki}]_+} \prod_{m=1}^{|b_{ki}|} \left(1 + q^{-\mathrm{sgn}(b_{ki})(2m-1)} Y_k\right)^{-\mathrm{sgn}(b_{ki})} & i \neq k \end{cases}$$

μ_k on $\mathbf{Y}$ is decomposed into monomial part and dilog (automorphism) part in two $(+,-)$ ways so that the following diagram becomes commutative:

$$\begin{array}{ccc} Y_i \in \mathbb{F}(\mathbf{Y}) & \xrightarrow{\mu_k} & \mathbb{F}(\mathbf{Y}) \\ \downarrow & & \uparrow \mu^\sharp_{k,\pm} \text{ dilog part} \\ Y_i' \in \mathbb{F}(\mathbf{Y}') & \xrightarrow[\tau_{k,\pm}]{} & \mathbb{F}(\mathbf{Y}) \\ & \text{monomial part} & \end{array}$$

$$\tau_{k,\varepsilon}(Y_i') = q^{b_{ki}[\varepsilon b_{ik}]_+} Y_i Y_k^{[\varepsilon b_{ik}]_+} \quad (\varepsilon = \pm : \textit{sign})$$

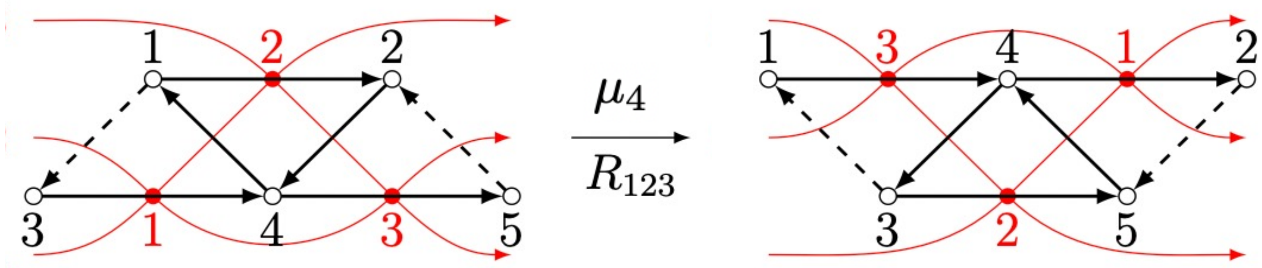
$$\mu^\sharp_{k,\varepsilon} = \mathrm{Ad}\big(\Psi_q(Y_k^\varepsilon)^\varepsilon\big), \ \text{ i.e. } \ \mu^\sharp_{k,\varepsilon}(Y_i) = \Psi_q(Y_k^\varepsilon)^\varepsilon Y_i \Psi_q(Y_k^\varepsilon)^{-\varepsilon}$$

Compositions of $\mu_k^* := \mathrm{Ad}\big(\Psi_q(Y_k^\varepsilon)^\varepsilon\big)\tau_{k,\varepsilon} : \mathbb{F}(\mathbf{Y}') \to \mathbb{F}(\mathbf{Y})$ are called cluster transformations.

Example $\quad \overset{1}{\underset{Y_1}{\circ}} \longrightarrow \overset{2}{\underset{Y_2}{\circ}} \quad \xrightarrow{\mu_2} \quad \overset{1}{\underset{Y_1(1+qY_2^{-1})^{-1}}{\circ}} \longleftarrow \overset{2}{\underset{Y_2^{-1}}{\circ}} \qquad b_{12} = 1 = -b_{21},\ Y_1Y_2 = q^2Y_2Y_1$

$$\begin{array}{lllll} & \overset{\tau_{2,+}}{\nearrow} & q^{-1}Y_1Y_2 & \xrightarrow{\mu^\sharp_{2,+}} & q^{-1}Y_1\Psi_q(q^{-2}Y_2)\Psi_q(Y_2)^{-1}Y_2 = q^{-1}Y_1(1+q^{-1}Y_2)^{-1}Y_2 \\ Y_1' & & & & \qquad \| \qquad Y_1(1+qY_2^{-1})^{-1} \\ & \underset{\tau_{2,-}}{\searrow} & Y_1 & \xrightarrow{\mu^\sharp_{2,-}} & \Psi_q(Y_2^{-1})^{-1}Y_1\Psi_q(Y_2^{-1}) = Y_1\Psi_q(q^2Y_2^{-1})^{-1}\Psi_q(Y_2^{-1}) \qquad \| \end{array}$$

Wiring diagrams (red) and the Fock-Goncharov (FG) quivers (black): Type A_2



$$(Y_1, \ldots, Y_5) \xleftarrow{\mu_4^*} (Y_1', \ldots, Y_5')$$

FG quivers are designed in such a way that the braid move R_{123} and the mutation μ_4 are compatible.

$$\mu_4^* : \begin{pmatrix} Y_1' \\ Y_2' \\ Y_3' \\ Y_4' \\ Y_5' \end{pmatrix} \overset{\tau_{4,+}}{\mapsto} \begin{pmatrix} Y_1 \\ q^{-1}Y_2Y_4 \\ q^{-1}Y_3Y_4 \\ Y_4^{-1} \\ Y_5 \end{pmatrix} \overset{\mathrm{Ad}(\Psi_q(Y_4))}{\mapsto} \begin{pmatrix} Y_1(1+qY_4) \\ Y_2(1+qY_4^{-1})^{-1} \\ Y_3(1+qY_4^{-1})^{-1} \\ Y_4^{-1} \\ Y_5(1+qY_4) \end{pmatrix}$$

The above transformation R_{123} of the wiring diagram satisfies the tetrahedron equation:

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}$$

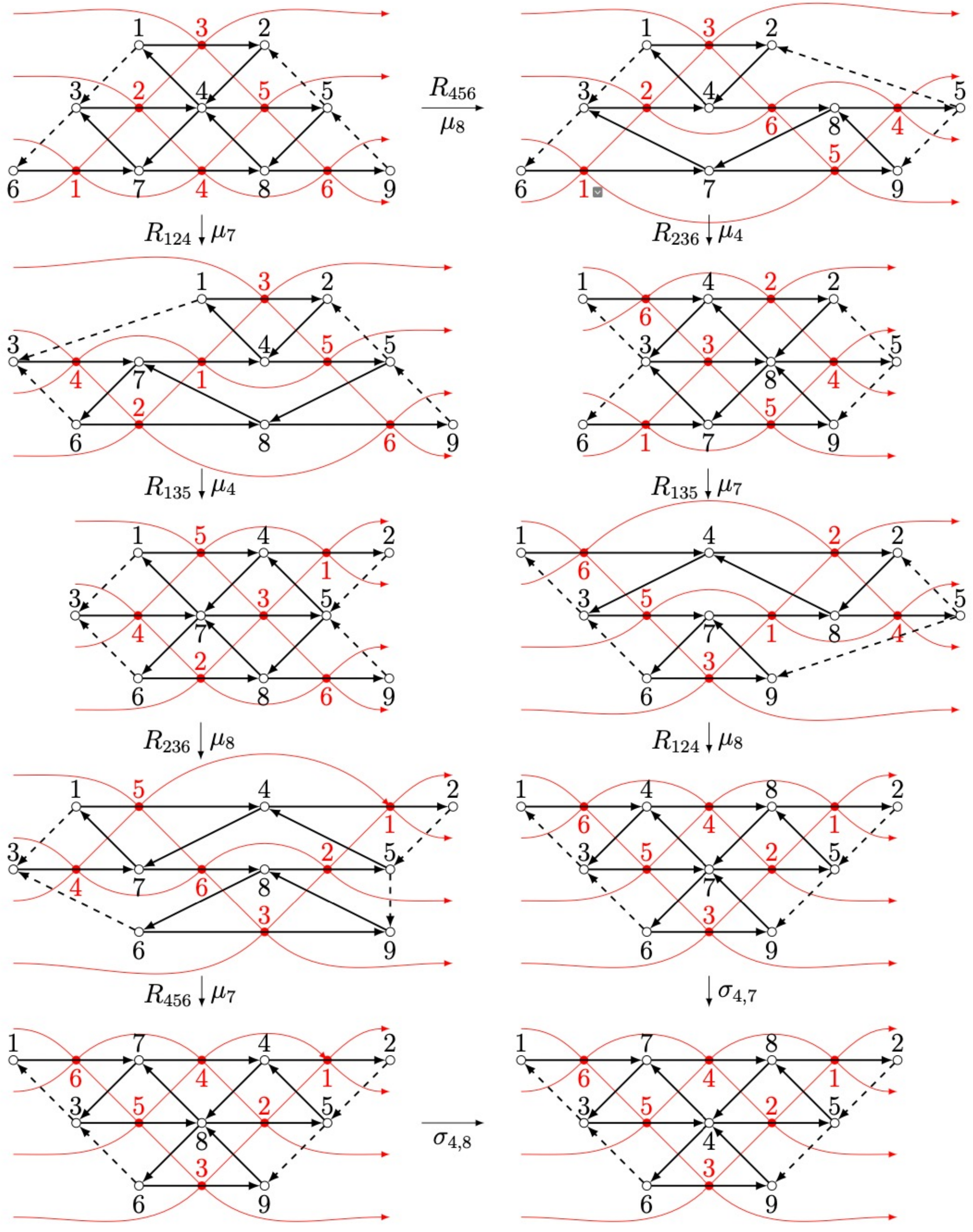
$A_2 \hookrightarrow A_3$

Wiring diagrams (red) which are successively transformed by braid moves denoted by R_{ijk}

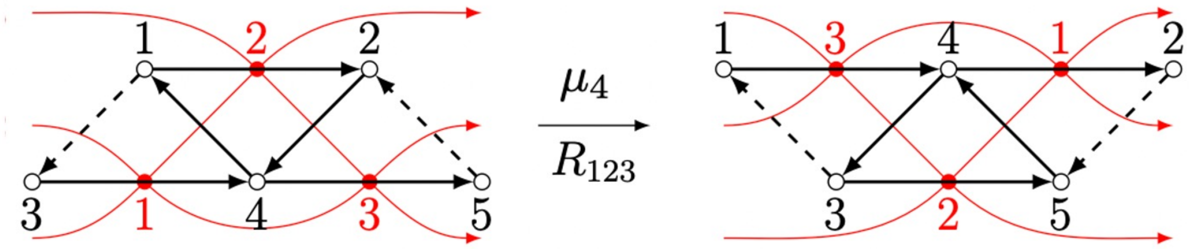
They are associated with the FG quivers (black) which are transformed by mutations μ_r

The figure shows that R_{ijk} satisfies the tetrahedron equation.

This is not so surprising.
Our upcoming main theorem is a further step beyond it.



Embedding into q-Weyl algebras



$$\xleftarrow{\mu_4^*}$$

$(Y_1, \dots, Y_5)$ $\qquad$ $(Y'_1, \dots, Y'_5)$

$$Y_1 Y_2 = q^2 Y_2 Y_1$$
$$Y_1 Y_3 = q Y_3 Y_1$$
$$Y_1 Y_4 = q^{-2} Y_4 Y_1$$
$$Y_1 Y_5 = Y_5 Y_1, \text{ etc}$$

$$Y'_1 Y'_2 = Y'_2 Y'_1$$
$$Y'_1 Y'_3 = q^{-1} Y'_3 Y'_1$$
$$Y'_1 Y'_4 = q^2 Y'_4 Y'_1$$
$$Y'_1 Y'_5 = Y'_5 Y'_1, \text{ etc}$$

The q-commutativity becomes automatic in the following parameterization using q-Weyl algebra

Introduce canonical variables:

$$[p_i, u_j] = \hbar \delta_{ij}, \quad [p_i, p_j] = [u_i, u_j] = 0$$

$e^{\pm p_i}, e^{\pm u_i}$ are generators of q-Weyl algebra

with the relation $e^{p_i} e^{u_j} = q^{\delta_{ij}} e^{u_j} e^{p_i}$

$$(q = e^{\hbar}, \quad \kappa_j = e^{\lambda_j}, \quad \lambda_{ij} = \lambda_i - \lambda_j)$$

$$Y_1 = \kappa_2^{-1} e^{p_2 - u_2 - p_1} \qquad Y'_1 = \kappa_3^{-1} e^{p_3 - u_3}$$
$$Y_2 = \kappa_2 e^{p_2 + u_2 - p_3} \qquad Y'_2 = \kappa_1 e^{p_1 + u_1}$$
$$Y_3 = \kappa_1^{-1} e^{p_1 - u_1} \qquad Y'_3 = \kappa_2^{-1} e^{p_2 - u_2 - p_3}$$
$$Y_4 = \kappa_1 \kappa_3^{-1} e^{p_1 + u_1 + p_3 - u_3 - p_2} \qquad Y'_4 = \kappa_1^{-1} \kappa_3 e^{p_3 + u_3 + p_1 - u_1 - p_2}$$
$$Y_5 = \kappa_3 e^{p_3 + u_3} \qquad Y'_5 = \kappa_2 e^{p_2 + u_2 - p_1}$$

Moreover, in the q-Weyl algebra, not only the dilogarithm part but also the monomial part of the cluster transformation

$$\begin{pmatrix} Y_1' \\ Y_2' \\ Y_3' \\ Y_4' \\ Y_5' \end{pmatrix} \overset{\tau_{4,+}}{\mapsto} \begin{pmatrix} Y_1 \\ q^{-1}Y_2Y_4 \\ q^{-1}Y_3Y_4 \\ Y_4^{-1} \\ Y_5 \end{pmatrix}$$

is realized as an adjoint as

$$\tau_{4,+} = \mathrm{Ad}(P_{123})$$

$$P_{123} = \rho_{23}\, e^{\frac{1}{\hbar}p_1(u_3-u_2)} e^{\frac{\lambda_{23}}{\hbar}(u_3-u_1)}$$

Example

$$\begin{aligned}
\mathrm{Ad}(P_{123})(e^{p_3}) &= \rho_{23}\, e^{\frac{1}{\hbar}p_1(u_3-u_2)} \underline{e^{\frac{\lambda_{23}}{\hbar}(u_3-u_1)} e^{p_3} e^{-\frac{\lambda_{23}}{\hbar}(u_3-u_1)}} e^{-\frac{1}{\hbar}p_1(u_3-u_2)} \rho_{23} \\
&= \rho_{23}\, \underline{e^{\frac{1}{\hbar}p_1(u_3-u_2)}} e^{-\lambda_{23}} \underline{e^{p_3} e^{-\frac{1}{\hbar}p_1(u_3-u_2)}} \rho_{23} \\
&= \rho_{23}\, e^{-p_1-\lambda_{23}} e^{p_3} \rho_{23} = e^{p_2-p_1-\lambda_{23}}.
\end{aligned}$$

Underlined parts are treated by the Baker-Campbell-Hausdorff formula

Therefore, the cluster transformation μ_4^* becomes totally an adjoint as

$$\mu_4^* = \mathrm{Ad}(\Psi_q(Y_4))\tau_{4,+} = \mathrm{Ad}(\Psi_q(Y_4))\mathrm{Ad}(P_{123}) = \mathrm{Ad}(\mathcal{R}_{123})$$

$$\mathcal{R}_{123} = \Psi_q(Y_4)P_{123} = \Psi_q(e^{p_1+u_1+p_3-u_3-p_2+\lambda_{13}})\rho_{23}e^{\frac{1}{\hbar}p_1(u_3-u_2)}e^{\frac{\lambda_{23}}{\hbar}(u_3-u_1)}$$

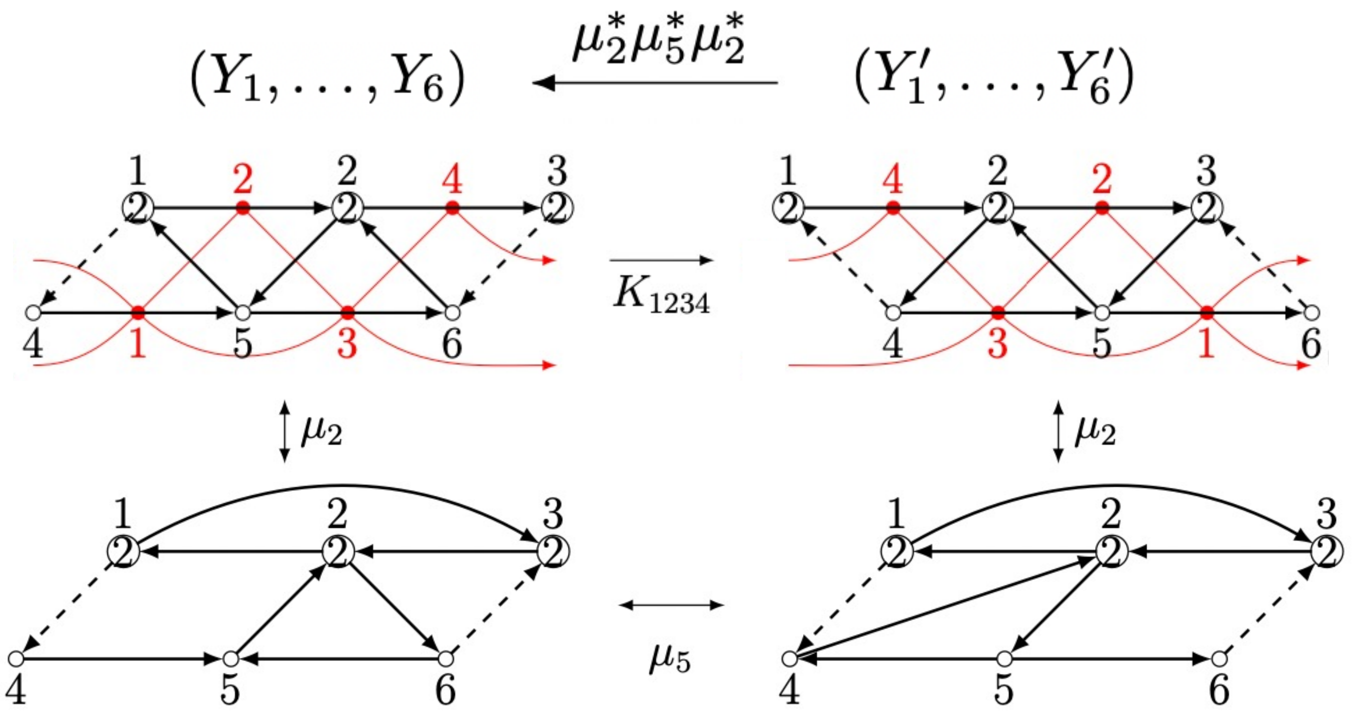
$$= \mathcal{R}(\lambda_1, \lambda_2, \lambda_3)_{123}$$

Theorem. The tetrahedron equation with spectral parameters is valid:

$$\mathcal{R}(\lambda_4, \lambda_5, \lambda_6)_{456}\mathcal{R}(\lambda_2, \lambda_3, \lambda_6)_{236}\mathcal{R}(\lambda_1, \lambda_3, \lambda_5)_{135}\mathcal{R}(\lambda_1, \lambda_2, \lambda_4)_{124}$$
$$= \mathcal{R}(\lambda_1, \lambda_2, \lambda_4)_{124}\mathcal{R}(\lambda_1, \lambda_3, \lambda_5)_{135}\mathcal{R}(\lambda_2, \lambda_3, \lambda_6)_{236}\mathcal{R}(\lambda_4, \lambda_5, \lambda_6)_{456}$$

Wiring diagrams (red) and the FG quivers (black) for K : Type C_2

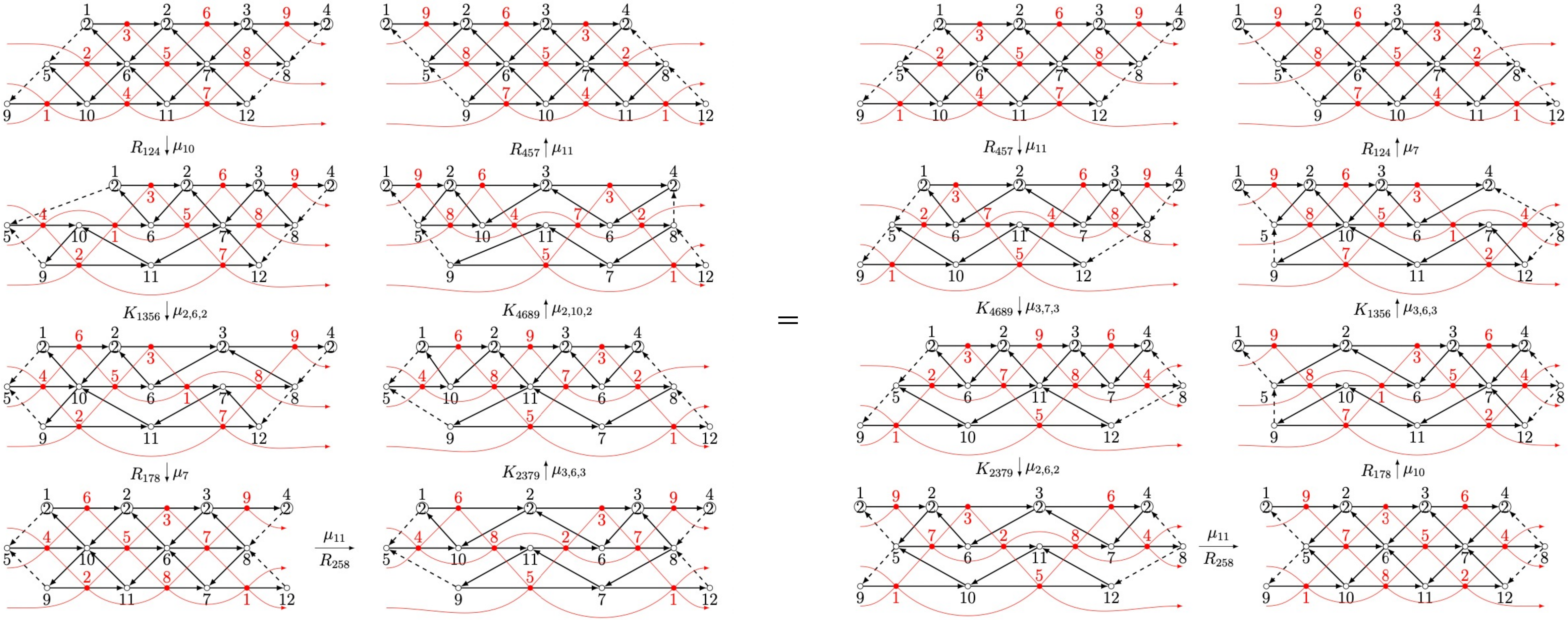
FG quivers are *weighted*. (②= weight 2 node, Exchange matrices are only skew-*symmetrizable*)



The transformation K_{1234} of the wiring diagram induces the following cluster transformation:

$$\mu_2^*\,\mu_5^*\,\mu_2^* = \mathrm{Ad}(\Psi_{q^2}(Y_2)\Psi_q(Y_5)\Psi_{q^2}(Y_2)^{-1})\tau_{2,+}\tau_{5,+}\tau_{2,-}$$

For three reflecting wires (red), there are two ways to reverse the order of reflections: $C_2 \hookrightarrow C_3$



The corresponding transformations K and R satisfy the 3D reflection equation:

$$R_{457}K_{4689}K_{2379}R_{258}R_{178}K_{1356}R_{124} = R_{124}K_{1356}R_{178}R_{258}K_{2379}K_{4689}R_{457}.$$

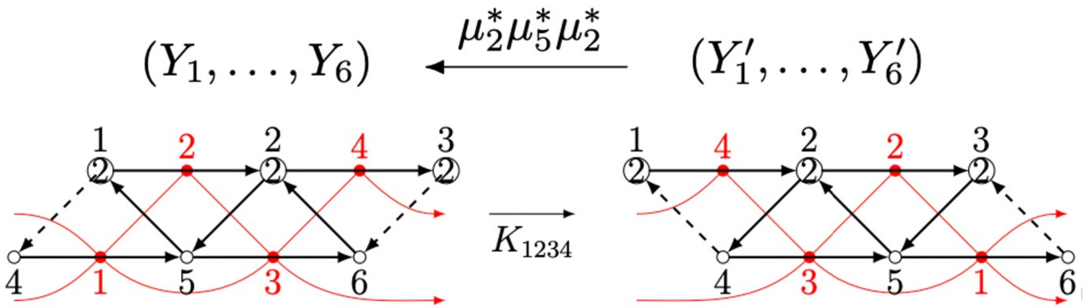
The cluster transformation induced by K_{1234}

$$\mu_2^*\mu_5^*\mu_2^* : \begin{pmatrix} Y_1' \\ Y_2' \\ Y_3' \\ Y_4' \\ Y_5' \\ Y_6' \end{pmatrix} \xmapsto{\tau_{2,+}\tau_{5,+}\tau_{2,-}} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ q^{-1}Y_4Y_5 \\ q^2Y_5^{-1}Y_2^{-1} \\ q^{-1}Y_2Y_5Y_6 \end{pmatrix} \xmapsto{\mathrm{Ad}(\Psi_{q^2}(Y_2)\Psi_q(Y_5)\Psi_{q^2}(Y_2)^{-1})} \begin{pmatrix} Y_1\Lambda_0 \\ \Lambda_1^{-1}\Lambda_2^{-1}Y_2 \\ \Lambda_0^{-1}Y_3\Lambda_1\Lambda_2 \\ q^{-1}\Lambda_0^{-1}Y_4Y_5\Lambda_1 \\ q^2Y_5^{-1}Y_2^{-1}\Lambda_0 \\ q^{-1}\Lambda_1^{-1}Y_2Y_5Y_6 \end{pmatrix}$$

$$\Lambda_0 = 1 + (q+q^3)Y_5 + q^4Y_5^2(1+q^2Y_2), \quad \Lambda_1 = 1 + qY_5(1+q^2Y_2), \quad \Lambda_2 = 1 + q^3Y_5(1+q^2Y_2)$$

An embedding of Y-variables into q-Weyl algebras (p_i and u_i obey the canonical commutation relation)

$$\begin{aligned} Y_1 &\mapsto \kappa_2^{-1}e^{p_2-u_2-2p_1}, \\ Y_2 &\mapsto \kappa_2\kappa_4^{-1}e^{p_2+u_2+p_4-u_4-2p_3}, \\ Y_3 &\mapsto \kappa_4\, e^{p_4+u_4}, \\ Y_4 &\mapsto \kappa_1^{-1}e^{p_1-u_1}, \\ Y_5 &\mapsto \kappa_1\kappa_3^{-1}e^{p_1+u_1+p_3-u_3-p_2}, \\ Y_6 &\mapsto \kappa_3\, e^{p_3+u_3-p_4}, \end{aligned}$$



$$\begin{aligned} Y_1' &\mapsto \kappa_4^{-1}e^{p_4-u_4}, \\ Y_2' &\mapsto \kappa_4\kappa_2^{-1}e^{p_4+u_4+p_2-u_2-2p_3}, \\ Y_3' &\mapsto \kappa_2\, e^{p_2+u_2-2p_1}, \\ Y_4' &\mapsto \kappa_3^{-1}e^{p_3-u_3-p_4}, \\ Y_5' &\mapsto \kappa_3\kappa_1^{-1}e^{p_3+u_3+p_1-u_1-p_2}, \\ Y_6' &\mapsto \kappa_1\, e^{p_1+u_1}. \end{aligned}$$

Under this embedding, the cluster transformation for K_{1234} becomes totally an adjoint as

$$\mu_2^* \mu_5^* \mu_2^* = \mathrm{Ad}(\mathcal{K}_{1234})$$

$$\begin{aligned}\mathcal{K}_{1234} &= \mathcal{K}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)_{1234} \\ &= \Psi_{q^2}(e^{p_2+u_2+p_4-u_4-2p_3+\lambda_{24}}) \Psi_q(e^{p_1+u_1+p_3-u_3-p_2+\lambda_{13}}) \Psi_{q^2}(e^{p_2+u_2+p_4-u_4-2p_3+\lambda_{24}})^{-1} \\ &\quad \times \rho_{24}\, e^{\frac{1}{\hbar} p_1 (u_4-u_2)} e^{\frac{\lambda_{24}}{2\hbar}(2u_3-2u_1+u_4-u_2)}\end{aligned}$$

Theorem. The 3D reflection equation with spectral parameters is valid:

$$\mathcal{R}_{457}\mathcal{K}_{4689}\mathcal{K}_{2379}\mathcal{R}_{258}\mathcal{R}_{178}\mathcal{K}_{1356}\mathcal{R}_{124} = \mathcal{R}_{124}\mathcal{K}_{1356}\mathcal{R}_{178}\mathcal{R}_{258}\mathcal{K}_{2379}\mathcal{K}_{4689}\mathcal{R}_{457}$$

where $\mathcal{R}_{ijk} = \mathcal{R}(\lambda_i, \lambda_j, \lambda_k)_{ijk}$ and $\mathcal{K}_{ijkl} = \mathcal{K}(\lambda_i, \lambda_j, \lambda_k, \lambda_l)_{ijkl}$.

4. Tetrahedron equality as duality

A representation of the q-Weyl algebra $e^{p_i} e^{u_j} = q^{2\delta_{ij}} e^{u_j} e^{p_i}$ on $\bigoplus_{m_1,m_2,m_3 \in \mathbb{Z}^3} \mathbb{C}|m_1,m_2,m_3\rangle$

$$e^{p_i}|m_1,m_2,m_3\rangle = |m_1,m_2,m_3\rangle|_{m_i \to m_i - 1}, \quad e^{u_i}|m_1,m_2,m_3\rangle = q^{2m_i}|m_1,m_2,m_3\rangle$$

Matrix elements : $R^{a,b,c}_{i,j,k} := \langle a,b,c|\mathcal{R}_{123}|i,j,k\rangle = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\left(-\frac{\kappa_1}{\kappa_3}\right)^{b-k}\left(\frac{\kappa_2}{\kappa_3}\right)^{k-i}\frac{q^{(b-k)(i-k+1)}}{(q^2;q^2)_{b-k}}$

Substitution of this into the tetrahedron equality

$$\begin{aligned}&\sum_{b_1,\ldots,b_6 \in \mathbb{Z}} R^{a_1,a_2,a_4}_{b_1,b_2,b_4}(\lambda_1,\lambda_2,\lambda_4) R^{b_1,a_3,a_5}_{c_1,b_3,b_5}(\lambda_1,\lambda_3,\lambda_5) R^{b_2,b_3,a_6}_{c_2,c_3,b_6}(\lambda_2,\lambda_3,\lambda_6) R^{b_4,b_5,b_6}_{c_4,c_5,c_6}(\lambda_4,\lambda_5,\lambda_6) \\ &= \sum_{b_1,\ldots,b_6 \in \mathbb{Z}} R^{a_4,a_5,a_6}_{b_4,b_5,b_6}(\lambda_4,\lambda_5,\lambda_6) R^{a_2,a_3,b_6}_{b_2,b_3,c_6}(\lambda_2,\lambda_3,\lambda_6) R^{a_1,b_3,b_5}_{b_1,c_3,c_5}(\lambda_1,\lambda_3,\lambda_5) R^{b_1,b_2,b_4}_{c_1,c_2,c_4}(\lambda_1,\lambda_2,\lambda_4).\end{aligned}$$

is distilled into the *duality* of q-series under the interchange $r \longleftrightarrow s$:

$$\frac{1}{(q^2)_{s+t}} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n(n+1+2s)}}{(q^2)_n (q^2)_{t-n} (q^2)_{n+r}} = \frac{1}{(q^2)_{r+t}} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n(n+1+2r)}}{(q^2)_n (q^2)_{t-n} (q^2)_{n+s}}$$

5. Outlook

A similar duality is present also in the *modular double* setting,
where the matrix elements involve non-compact quantum dilogarithm (NCQD).

$$\varphi(z) = \exp\left(\frac{1}{4}\int \frac{e^{-2izw}}{\sinh(w\mathfrak{b})\sinh(w/\mathfrak{b})}\frac{dw}{w}\right) \qquad q = e^{i\pi\mathfrak{b}^2}$$

The duality in that case emerges as an identity of integrals, which is also reproduced by a NCQD analogue of a classical Heine transformation.

Future problems:

- Possible connections with the duality in 3D supersymmetric gauge theories
- Exploring insights into 3D consistency
- Butterfly quiver
- Reductions to 2D, $q^N = 1,\ q \to 0,$ etc.

感谢您的关注

Thank you